Vortex state in a Bose-Fermi mixture with attraction between bosons and fermions

A. M. Belemuk, N. M. Chtchelkatchev, and V. N. Ryzhov
Institute for High Pressure Physics, Russian Academy of Sciences, Troitsk 142190, Moscow Region, Russia

S.-T. Chui
Bartol Research Institute, University of Delaware, Newark, Delaware 19716
(Received 3 March 2006; published 11 May 2006)

An effective Hamiltonian for the Bose subsystem for the mixture of ultracold atomic clouds of bosons and fermions is obtained by integrating out the fermion degrees of freedom. Using the effective Hamiltonian, a collapse of the trapped boson-fermion mixture due to the boson-fermion attractive interaction without and in the presence of the quantized vortices is studied in the framework of variational Bose wave function and Thomas-Fermi approximation. The properties of the ^{87}Rb and ^{40}K mixture are analyzed quantitatively. The critical number of bosons for the collapse transition is estimated without and in the presence of the vortices as a function of the fermion number. It is shown that the critical number of bosons increases in the presence of the vortex. The vortex critical velocities are calculated as functions of the numbers of bosons and fermions.

DOI: 10.1103/PhysRevA.73.053608 PACS number(s): 03.75.Lm, 03.75.Kk, 67.57.Fg

I. INTRODUCTION

Since the first realization of Bose-Einstein condensation (BEC) in ultracold atomic gas clouds [1–3], studies in this direction have yielded unprecedented insight into the quantum statistical properties of matter. In addition to the studies using the bosonic atoms, growing interest is focused on the cooling of fermionic atoms to a temperature regime where quantum effects dominate the properties of the gas [4–6]. This interest is mainly motivated by the quest for the crossover between a BEC and Bardeen-Cooper-Schrieffer (BCS) superfluid in ultarcold atomic Fermi gasses [7–9].

Strong s-wave interactions that facilitate evaporative cooling of bosons are absent among spin-polarized fermions due to the exclusion Pauli principle. So the fermions are cooled to degeneracy through the mediation of fermions in another spin state [4,6–9] or via a buffer gas of bosons [5,10,11] (sympathetic cooling). The Bose gas, which can be cooled evaporatively, is used as a coolant, the fermionic system being in thermal equilibrium with the cold Bose gas through boson-fermion interaction in the region of overlapping of the systems.

However, the physical properties of Bose-Fermi mixtures are interesting in their own rights and are the subject of intensive investigations including the analysis of ground state properties, stability, effective Fermi-Fermi interaction mediated by the bosons, and new quantum phases in optical lattices [12–18]. Several successful attempts to trap and cool mixtures of bosons and fermions were reported. Quantum degeneracy was first reached with mixtures of bosonic ^{7}Li and fermionic ^{6}Li atoms [5,10]. Later, experiments to cool mixtures of ^{23}Na and ^{6}Li [19], as well as ^{87}Rb and ^{40}K [11,21], to ultralow temperatures succeeded.

Although Bose-Einstein condensation and supefluidity are closely connected, they do not occur together in all cases. For example, in lower dimensions one can observe superfluid without BEC. In principle, rotational properties of a Bose-Fermi mixture could directly reveal superfluidity in such systems. Quantized vortices in a rotating gas provide direct conclusive evidence for superfluidity because they are a consequence of of the existence of a macroscopic wave function that describes the superfluid. Recently such experimental evidence was obtained for the superfluidity in strongly interacting Fermi gas [20].

In this article we study the instability and collapses of the trapped boson-fermion mixture due to the boson-fermion attractive interaction in the presence of the quantized vortices, using the effective Hamiltonian for the Bose system [14,15]. We analyze quantitatively properties of the ^{87}Rb and ^{40}K mixture with an attractive interaction between bosons and fermions. The stability of this system without vortices was recently studied [15], and good agreement with experiment by Modugno and co-workers [11] was found. As was shown in the experiment [11], as the number of bosons is increased there is an instability value N_{Bc} at which a discontinues leakage of the bosons and fermions occurs, and collapse of boson and fermion clouds is observed. In this article we estimated the instability boson number N_{Bc} for the collapse transition in the presence of the vortices as a function of the fermion number and calculated the vortex critical velocities as functions of the numbers of bosons and fermions.

II. EFFECTIVE BOSE HAMILTONIAN

First of all we briefly discuss the effective boson Hamiltonian [14,15]. Our starting point is the functional-integral representation of the grand-canonical partition function of the Bose-Fermi mixture. It has the form [16,22,23]:

$$Z = \int D[\phi^*]D[\phi]D[\psi^*]D[\psi]\exp\left\{ -\frac{1}{\hbar}(S_B(\phi^*,\phi) + S_F(\psi^*,\psi) + S_{\text{int}}(\phi^*,\phi,\psi^*,\psi))\right\}. \quad (1)$$

and consists of an integration over a complex field $\phi(\tau,\mathbf{r})$, which is periodic on the imaginary-time interval $[0,\hbar\beta]$, and over the Grassmann field $\psi(\tau,\mathbf{r})$, which is antiperiodic on this interval. Therefore, $\phi(\tau,\mathbf{r})$ describes the Bose component of the mixture, whereas $\psi(\tau,\mathbf{r})$ corresponds to the Fermi component. The term describing the Bose gas has the form

$$S_B(\phi^*,\phi)=\int_0^{\hbar\beta}d\tau\int d\mathbf{r}\left\{\phi^*(\tau,\mathbf{r})\left(\hbar\frac{\partial}{\partial\tau}-\frac{\hbar^2\nabla^2}{2m_B}+V_B(\mathbf{r})-\mu_B\right)\phi(\tau,\mathbf{r})+\frac{g_B}{2}|\phi(\tau,\mathbf{r})|^4\right\}. \quad (2)$$

Because the Pauli principle forbids s-wave scattering between fermionic atoms in the same hyperfine state, the Fermi-gas term can be written in the form

$$S_F(\psi^*,\psi)=\int_0^{\hbar\beta}d\tau\int d\mathbf{r}\left\{\psi^*(\tau,\mathbf{r})\left(\hbar\frac{\partial}{\partial\tau}-\frac{\hbar^2\nabla^2}{2m_F}+V_F(\mathbf{r})-\mu_F\right)\psi(\tau,\mathbf{r})\right\}. \quad (3)$$

The term describing the interaction between the two components of the Fermi-Bose mixture is:

$$S_{\text{int}}(\phi^*,\phi,\psi^*,\psi)=g_{BF}\int_0^{\hbar\beta}d\tau\int d\mathbf{r}|\psi(\tau,\mathbf{r})|^2|\phi(\tau,\mathbf{r})|^2, \quad (4)$$

where $g_B=4\pi\hbar^2a_B/m_B$ and $g_{BF}=2\pi\hbar^2a_{BF}/m_I$, $m_I=m_Bm_F/(m_B+m_F)$, m_B and m_F are the masses of bosonic and fermionic atoms, respectively, a_B and a_{BF} are the s wave scattering lengths of boson-boson and boson-fermion interactions.

The integral over Fermi fields is Gaussian, we can calculate this integral and obtain the partition function of the Fermi system as a functional of Bose field $\phi(\tau,\mathbf{r})$. Let us rewrite the integral over $\psi(\tau,\mathbf{r})$ in the form

$$Z_F=\int D[\psi^*]D[\psi]\exp\left(-\frac{1}{\hbar}[S_F(\psi^*,\psi)+S_{\text{int}}(\phi^*,\phi,\psi^*,\psi)]\right)=\int D[\psi^*]D[\psi]\exp\left\{\int_0^{\hbar\beta}d\tau\int d\mathbf{r}\int_0^{\hbar\beta}d\tau'\int d\mathbf{r}'\times\psi^*(\tau,\mathbf{r})\mathbf{G}^{-1}(\tau,\mathbf{r},\tau',\mathbf{r}')\psi(\tau',\mathbf{r}')\right\}, \quad (5)$$

where

$$\mathbf{G}^{-1}=\mathbf{G}_0^{-1}-\Sigma \quad (6)$$

is the Dyson equation and $\Sigma(\tau,\mathbf{r},\tau',\mathbf{r}')$ is a self-energy:

$$\mathbf{G}_0^{-1}(\tau,\mathbf{r},\tau',\mathbf{r}')=-\frac{1}{\hbar}\left(\hbar\frac{\partial}{\partial\tau}-\frac{\hbar^2\nabla^2}{2m_F}+V_F(\mathbf{r})-\mu_F\right)\times\delta(\mathbf{r}-\mathbf{r}')\delta(\tau-\tau'); \quad (7)$$

$$\Sigma(\tau,\mathbf{r},\tau',\mathbf{r}')=\frac{g_{BF}}{\hbar}|\phi(\tau,\mathbf{r})|^2\delta(\mathbf{r}-\mathbf{r}')\delta(\tau-\tau'). \quad (8)$$

Using the formula for the gaussian integral over the Grassmann variables [22,23]

$$\int\prod_n d\psi_n^*d\psi_n\exp\left\{-\sum_{n,n'}\psi_n^*A_{n,n'}\psi_{n'}\right\}=e^{\text{Sp}[\ln A]} \quad (9)$$

one has

$$Z_F=\exp[\text{Sp}\ln(-\mathbf{G}^{-1})]=\exp\left(-\frac{1}{\hbar}S_{\text{eff}}\right). \quad (10)$$

S_{eff} can be expanded in powers of $|\phi(\tau,\mathbf{r})|^2$ by using the series

$$\mathbf{G}^{-1}=\mathbf{G}_0^{-1}-\Sigma=\mathbf{G}_0^{-1}(\mathbf{I}-\mathbf{G}_0\Sigma),$$

$$\text{Sp}[\ln(-\mathbf{G}^{-1})]=\text{Sp}[\ln(-\mathbf{G}_0^{-1})]-\sum_{n=1}^{\infty}\frac{1}{n}\text{Sp}[(\mathbf{G}_0\Sigma)^n]. \quad (11)$$

To proceed let us rewrite $\mathbf{G}_0$ in the form

$$\mathbf{G}_0(\tau,\mathbf{r},\tau',\mathbf{r}')=\sum_{\omega,n}\frac{-\hbar}{-i\hbar\omega+\epsilon_n-\mu_F}\xi_n(\mathbf{r})\xi_n^*(\mathbf{r}')\frac{e^{-i\omega(\tau-\tau')}}{\hbar\beta}, \quad (12)$$

where $\omega=\pi(2s+1)/\hbar\beta$; $s=0,\pm1,\ldots$, and

$$\left(-\frac{\hbar^2\nabla^2}{2m_F}+V_F(\mathbf{r})\right)\xi_n(\mathbf{r})=\epsilon_n\xi_n(\mathbf{r}).$$

Because of large number of fermionic atoms in the system one can use the semiclassical Thomas-Fermi approximation [24]:

$$\sum_n\xi_n(\mathbf{r})\xi_n^*(\mathbf{r})F(\epsilon_n)=\frac{1}{(2\pi\hbar)^3}\int d\mathbf{p}F(H_0(\mathbf{p},\mathbf{r})), \quad (13)$$

where $H_0(\mathbf{p},\mathbf{r})=p^2/(2m_F)+V_F(\mathbf{r})$ and $F(x)$ is an arbitrary function.

We suppose that all $|\phi(\tau_i,\mathbf{r}_i)|^2$ have one and the same argument $(\tau_1,\mathbf{r}_1)$ (see, for example, Ref. [22]). This assumption means that the fermions move faster than the bosons. Using Eq. (13), S_{eff} may be written in the form

$$S_{\text{eff}}=\int_0^{\hbar\beta}d\tau d\mathbf{r}f_{\text{eff}}[|\phi(\tau,\mathbf{r})|], \quad (14)$$

$$f_{\text{eff}}=-\frac{3}{2}\kappa\beta^{-1}\int_0^{\infty}\sqrt{\epsilon}d\epsilon\ln(1+e^{\beta(\tilde{\mu}-\epsilon)})=-\kappa\int_0^{\infty}\frac{\epsilon^{3/2}d\epsilon}{1+e^{\beta(\epsilon-\tilde{\mu})}}, \quad (15)$$

where $\epsilon=p^2/2m_F$, $\tilde{\mu}=\mu_F-V_F(\mathbf{r})-g_{BF}|\phi(\tau,\mathbf{r})|^2$, and $\kappa=2^{1/2}m_F^{3/2}/(3\pi^2\hbar^3)$. It should be noted that the semiclassical Thomas-Fermi approximation is valid if the number of particles is large and the variation of $\tilde{\mu}$ is small compared with the local Fermi energy [24].

Thus, the effective bosonic Hamiltonian may be written in the form

$$H_{\mathrm{eff}} = \int d\mathbf{r}\left\{ \frac{\hbar^2}{2m_B}|\nabla\phi|^2 + [V_B(\mathbf{r}) - \mu_B]|\phi|^2 + \frac{g_B}{2}|\phi|^4 + f_{\mathrm{eff}}(|\phi|)\right\}. \tag{16}$$

The first three terms in Eq. (16) have the conventional Gross-Pitaevskii [25] form, and the last term is a result of boson-fermion interaction. In the low temperature limit $\tilde{\mu}/(k_BT) \gg 1$ one can write $f_{\mathrm{eff}}(|\phi|)$ in the form

$$f_{\mathrm{eff}}(|\phi|) = -\frac{2}{5}\kappa\tilde{\mu}^{5/2} - \frac{\pi^2}{4}\kappa(k_BT)^2\tilde{\mu}^{1/2}. \tag{17}$$

As usual, μ_F can be determined from the equation

$$N_F = \int d\mathbf{r} n_F(\mathbf{r}), \tag{18}$$

where

$$n_F(\mathbf{r}) = \frac{3}{2}\kappa\int_0^\infty \frac{\sqrt{\epsilon}d\epsilon}{1 + e^{\beta(\epsilon-\tilde{\mu})}}. \tag{19}$$

At low temperatures we have

$$n_F(\mathbf{r}) = \kappa\tilde{\mu}^{3/2} + \frac{\pi^2\kappa}{8\tilde{\mu}^{1/2}}(k_BT)^2. \tag{20}$$

In the general case the Bose and the Fermi systems have different temperature scales, and Eqs. (17) and (20) may be useful for studying the temperature behavior of Bose system, including the calculation of the critical temperature. For example, the characteristic temperature for the Bose system—the transition temperature for the ideal Bose gas—is [25] $k_BT_c^0 = 0.94\hbar\omega_B(\lambda N_B)^{1/3}$. The Fermi temperature for a pure system is [24] $k_BT_F = \hbar\omega_F(6\lambda N_F)^{1/3}$. Taking into account that $\omega_F = \sqrt{m_B/m_F}\omega_B$, one can see that for $m_B > m_F$ and approximately the same numbers of bosons and fermions $T_F > T_c^0$; one can safely use Eqs. (17) and (20) to describe the behavior of the Bose system.

Let us consider now the ^{87}Rb and ^{40}K mixture with an attractive interaction between the bosons and the fermions [11]. The parameters of the system are the following: $a_B = 5.25$ nm, $a_{BF} = -21.7^{+4.3}_{-4.8}$ nm. K and Rb atoms were prepared in the doubly polarized states $|F=9/2, m_F=9/2\rangle$ and $|2, 2\rangle$, respectively. The magnetic potential had an elongated symmetry, with harmonic oscillation frequencies for Rb atoms $\omega_{B,r} = \omega_B = 2\pi \times 215$ Hz and $\omega_{B,z} = \lambda\omega_B = 2\pi \times 16.3$ Hz, $\omega_F = \sqrt{m_B/m_F}\omega_B \approx 1.47\omega_B$, so that $m_B\omega_B^2/2 = m_F\omega_F^2/2 = V_0$. The collapse was found for the following critical numbers of bosons and fermions: $N_{Bc} \approx 10^5$; $N_K \approx 2 \times 10^4$.

At the zero temperature limit, expanding $f_{\mathrm{eff}}(|\phi|)$ up to the third order in g_{BF} we obtain the effective Hamiltonian in the form

$$H_{\mathrm{eff}} = \int d\mathbf{r}\left\{ \frac{\hbar^2}{2m_B}|\nabla\phi|^2 + [V_{\mathrm{eff}}(\mathbf{r}) - \mu_B]|\phi|^2 + \frac{g_{\mathrm{eff}}}{2}|\phi|^4 + \frac{\kappa}{8\mu_F^{1/2}}g_{BF}^3|\phi|^6\right\}, \tag{21}$$

where

$$V_{\mathrm{eff}}(\mathbf{r}) = \left(1 - \frac{3}{2}\kappa\mu_F^{1/2}g_{BF}\right)\frac{1}{2}m_B\omega_B^2(\rho^2 + \lambda^2z^2), \tag{22}$$

$$g_{\mathrm{eff}} = g_B - \frac{3}{2}\kappa\mu_F^{1/2}g_{BF}^2, \tag{23}$$

and $\rho^2 = x^2 + y^2$.

In principle, one can study the properties of a Bose-Fermi mixture with the help of f_{eff} (17) without any expansion. However, the form of the Hamiltonian (21) gives the possibility to get a clear insight into the physics of the influence of the Fermi system on the Bose one (see discussion below). It may be easily verified that the expansion of the function $f(x) = (1+x)^{5/2}$ [see Eq. (17)] up to the third order in x gives a reasonably good approximation for $f(x)$ even for rather large values of x, in contrast with the higher order expansions, so one can safely use Eq. (21) as a starting point for the investigation of the properties of the Bose subsystem.

In the derivation of Eqs. (21)–(23) we also use the fact that due to the Pauli principle (quantum pressure) the radius of the Bose condensate is much less than the radius of the Fermi cloud $R_F \approx \sqrt{\mu_F/V_0}$, so one can use an expansions in powers of $V_F(\mathbf{r})/\mu_F$. From Eq. (22) one can see that the interaction with Fermi gas leads to modification of the trapping potential. For the attractive fermion-boson interaction the system should behave as if it was confined in a magnetic trapping potential with larger frequencies than the actual ones, in agreement with experiment [11]. Boson-fermion interaction also induces the additional attraction between Bose atoms which does not depend on the sign of g_{BF}.

The last term in H_{eff} (21) corresponds to the three-particle *elastic* collisions induced by the boson-fermion interaction. In contrast with *inelastic* three-body collisions which result in the recombination and removing particles from the system [26], this term for $g_{BF} < 0$ leads to increase of the gas density in the center of the trap in order to lower the total energy. The positive zero point energy and boson-boson repulsion energy [the first two terms in Eq. (21)] stabilize the system. However, if the central density grows too much, the kinetic energy and boson-boson repulsion are no longer able to prevent the collapse of the gas. Likewise the case of Bose condensate with attraction (see, for example, Refs. [25–27]), the collapse is expected to occur when the number of particles in the condensate exceeds the critical value N_{Bc}.

III. VARIATIONAL APPROACH

The qualitative (and even quantitative) picture of the stability of the system can be obtained in the framework of the variational approach. In the system without vortex the critical number N_{Bc} can be calculated using the well-known ansatz for the Bosonic wave function [25]

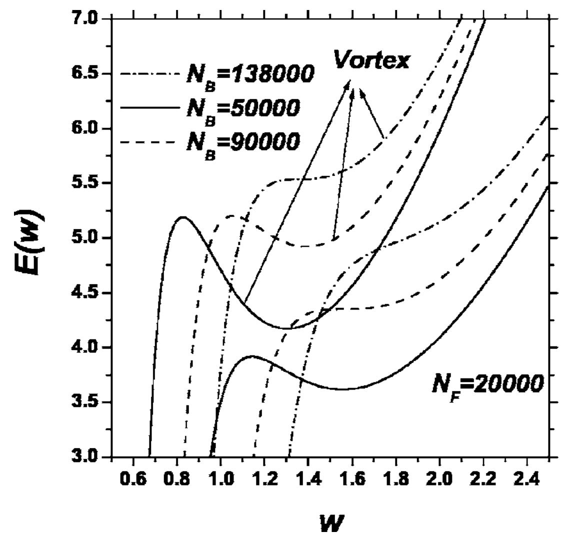

FIG. 1. Variational energy $E_B/N_B\hbar\omega_B$ as a function of w for various numbers of bosons. Upper lines correspond to the energy of the system with a vortex.

$$\phi(\mathbf{r}) = \left(\frac{N_B\lambda}{w^3a^3\pi^{3/2}}\right)^{1/2}\exp\left(-\frac{(\rho^2+\lambda^2z^2)}{2w^2a^2}\right), \qquad (24)$$

where w is a dimensionless variational parameter which fixes the width of the condensate and $a=\sqrt{\hbar/m_B\omega_B}$.

In this case the variational energy E_B has the form

$$\frac{E_B}{N_B\hbar\omega_B} = \frac{2+\lambda}{4}\frac{1}{w^2} + bw^2 + \frac{c_1N_B}{w^3} + \frac{c_2N_B^2}{w^6},$$

$$b = \frac{3}{4}\left(1-\frac{3}{2}\kappa\mu_F^{1/2}g_{BF}\right),$$

$$c_1 = \frac{1}{2}\left(g_B - \frac{3}{2}\kappa\mu_F^{1/2}g_{BF}^2\right)\frac{\lambda}{(2\pi)^{3/2}\hbar\omega_Ba^3},$$

$$c_2 = \frac{\kappa}{8\mu_F^{1/2}}g_{BF}^3\frac{\lambda^2}{3^{3/2}\pi^3\hbar\omega_Ba^6}. \qquad (25)$$

This energy is plotted in Fig. 1 as a function of w for several values of N_B. It is seen that when $N_B < N_{Bc}$ there is a local minimum of E_B which correspond to a metastable state of the system. This minimum arises due the competition between the positive first three terms in Eq. (25) and negative fourth term. The local minimum disappears when the number of bosons N_B exceeds the critical value which can be calculated by requiring that the first and second derivatives of E_B vanish at the critical point. In this case the behavior of E_B is mainly determined by the second and fourth terms in Eq. (25). For $N_K=2\times10^4$ and $a_{BF}=-19.44$ nm we obtain $N_{Bc}\approx 9\times10^4$ in a good agreement with the experiment [11]. It is interesting to note that the critical number of Bose atoms in Bose-Fermi mixture is about two orders larger than the critical number for the condensate with a purely attractive interaction. For example, in the experiments with trapped ^{7}Li [3] it was found that the critical number of bosons is about 1000.

If a vortex with angular momentum $\hbar l$ is present in the system, the variational wave function may be written in the form [28]:

$$\phi(\mathbf{r}) = \sqrt{\frac{\lambda N_B}{(\omega a)^5\pi^{3/2}}}\rho\exp\left(-\frac{\rho^2+\lambda^2z^2}{2\omega^2a^2}\right)e^{i\varphi l}. \qquad (26)$$

In this case the variational energy has the form

$$\frac{E_b}{N_B\hbar\omega_B} = \frac{2+\lambda^2+2l^2}{4}\frac{1}{\omega^2} + B\omega^2 + \frac{C_1N_B}{\omega^3} + \frac{C_2N_B^2}{\omega^6},$$

$$B = \frac{5}{3}b,$$

$$C_1 = \frac{1}{2}c_1,$$

$$C_2 = \frac{2}{9}c_2.$$

From Fig. 1 one can see that the stability of the system with a vortex is higher (in this case the critical number of bosons $N_{Bc}\approx 135\times10^3$) due to the centrifugal effect [compare the first terms in Eqs. (25) and (26) and decreasing the number density of the bosons in the center of the trap, fourth terms in Eqs. (25) and (26)].

In Eq. (25) we use $\mu_F^0=\hbar\omega_F[6\lambda N_F]^{1/3}$ as the chemical potential of the Fermi system μ_F. The corrections to μ_F due to interaction with the Bose system have the form $\mu_F=\mu_F^0[1+m_1]$, where $m_1=1/2\kappa g_{BF}\mu_F^{0^{1/2}}N_B/N_F$. It may be shown that $m_1\approx 0.09$ for the values of the parameters used in these calculations. The correction to the chemical potential for the vortex state is the same.

Upon increasing the number of fermions, the repulsion between bosons decreases leading to the collapse for the smaller numbers of the bosonic atoms. In Fig. 2 the critical number of bosons N_{Bc} is represented as a function of the number of fermions for both cases: the state without vortex and the state with the vortex.

It should be noted that the critical number of bosons N_{Bc} is extremely sensitive to the precise value of the boson-fermion s wave scattering length. This is illustrated in Fig. 3 in [29].

The critical velocity of the vortex state can be found from the formula [28]: $\Omega_c=[E_b(l=1)-E_b]/\hbar N_B$. In Figs. 4 and 5 Ω_c is shown as a function of N_B and N_F.

The curves in Figs. 3 and 4 are ending at the critical points which correspond to the critical numbers of bosons and fermions at which the collapse of the system occurs. The minima in Fig. 3 and intersections of the curves in Fig. 4 seem to be artifacts of variational approach to calculation of the critical angular velocity in the vicinity of the critical points. It should be noted that the number of bosons in the system is rather large. In this case the qualitatively and even quantitatively correct results may be obtained in the framework of the Thomas-Fermi approximation (TFA).

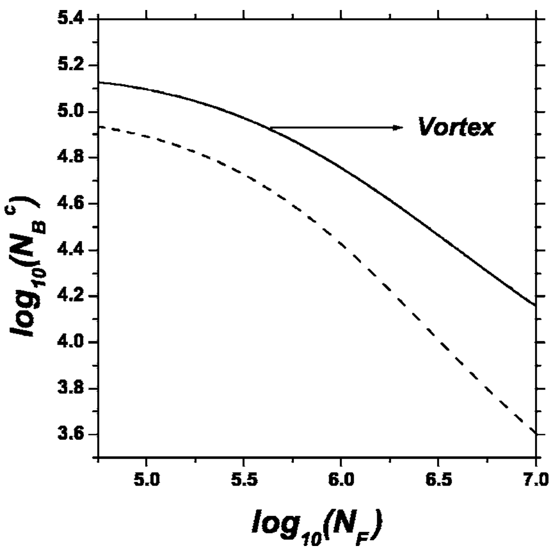

FIG. 2. Critical number of bosons N_{Bc} as a function of the number of fermions N_F at $T=0$.

IV. THOMAS-FERMI APPROXIMATION

In the Thomas-Fermi approximation the kinetic energy terms are ignored. It has been shown that in the case of one component condensates the TFA results agree well with the numerical calculations for large particle numbers, except for a small region near the boundary of the condensate [25,28,30]. In fact, even for a small number of particles the TFA still usually gives qualitatively correct results. The TFA provides an excellent starting point of study of the vortex states in Bose condensates (see, for example, Refs. [29,31,32]).

The Gross-Pitaevski equation that follows from the effective Hamiltonian (21):

$$\left(-\frac{\hbar^2}{2m_B}\Delta\phi+(V_{\rm eff}-\mu_B)+g_{\rm eff}|\phi|^2 \right. \tag{27}$$

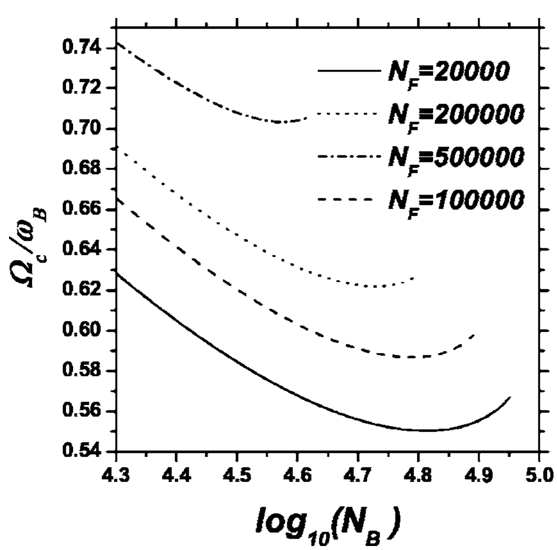

FIG. 3. Critical angular vortex velocity as a function of a logarithm of the number of bosons N_B.

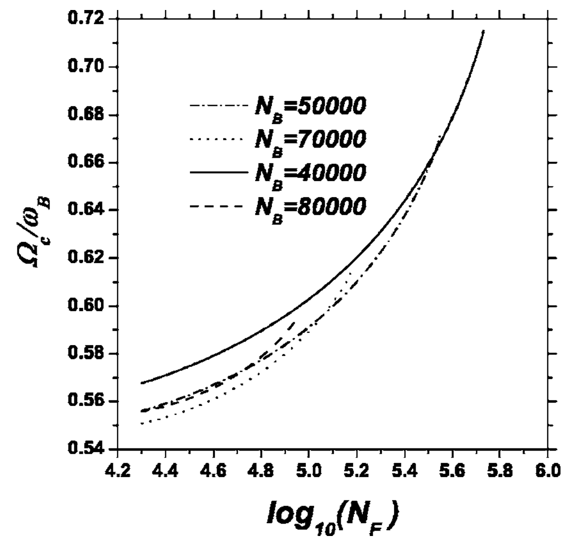

FIG. 4. Critical angular vortex velocity as a function of a logarithm of the number of fermions N_F.

$$\left. +\frac{3\kappa}{8\mu_F^{1/2}}g_{BF}^3|\phi|^4\right)\phi=0. \tag{28}$$

In TFA one can neglect the kinetic energy. In this case boson density has the form

$$|\phi|^2=\theta(\mu_B-V_{\rm eff})\frac{g_{\rm eff}\left(-1+\sqrt{1+(\mu_B-V_{\rm eff})\dfrac{3\kappa g_{BF}^3}{g_{\rm eff}^2 2\mu_F^{1/2}}}\right)}{\dfrac{3\kappa g_{BF}^3}{4\mu_F^{1/2}}}. \tag{29}$$

In the limit $g_{BF}\to 0$ one has the conventional expression for the boson density [25,28]

$$|\phi|^2=\theta(\mu_B-V_{\rm eff})\frac{(\mu_B-V_{\rm eff})}{g_{\rm eff}}.$$

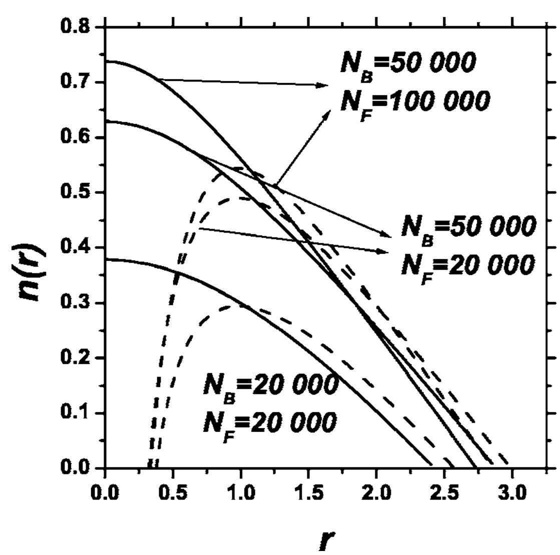

FIG. 5. Density distribution of bosons as a function of radius for different numbers of bosons and fermions calculated in TFA.

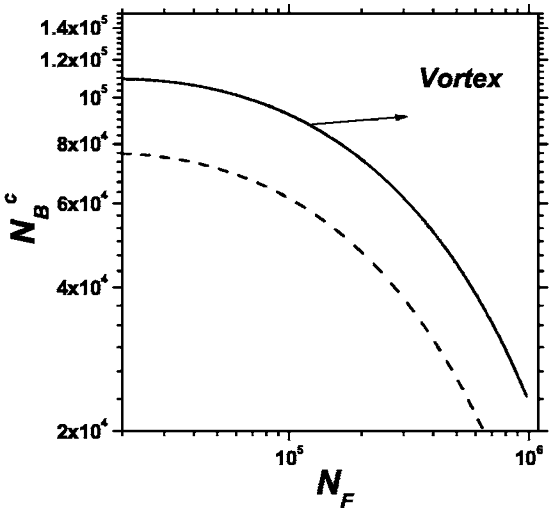

FIG. 6. Critical number of bosons N_{Bc} as a function of the number of fermions N_F in TFA.

Equation (33) for the boson density $n(\mathbf{r})=|\phi(\mathbf{r})|^2$ may be rewritten in the form

$$n(\mathbf{r}) = n(0)\left(1 - \sqrt{1 + \frac{\bar{x}^2 + \bar{y}^2 + \lambda^2\bar{z}^2 - R^2}{R_{\max}^2}}\right). \tag{30}$$

Here

$$\bar{r}_i = r_i/a_h; \quad R^2 = \frac{2\mu_B}{c_0 m_B \omega_B^2 a_h^2};$$

$$n(0) = -\frac{4}{3}\frac{g_{\text{eff}}\mu_F^{1/2}}{\kappa g_{BF}^3}; \quad R_{\max}^2 = -\frac{4}{3}\frac{g_{\text{eff}}^2\mu_F^{1/2}}{\kappa g_{BF}^3 c_0 m_B \omega_B^2 a_h^2};$$

$$c_0 = \left(1 - \frac{3}{2}\kappa\mu_F^{1/2} g_{BF}\right).$$

$R_{\max}$ is the radius of the condensate. The critical number of bosons can be determined from the condition that the expression under the square root in Eq. (30) is positive. In Fig. 5 the density distribution of bosons calculated from Eqs. (30) is shown as a function of radius for different numbers of bosons and fermions. In Fig. 6 the critical number of bosons is shown as a function of the number of fermions (lower curve).

Let us now consider a trap rotating with frequency Ω along the z axis. For vortex excitation with angular momentum $\hbar l$, the condensate wave function is given by

$$\phi_l(\mathbf{r}) = \sqrt{n_l(\mathbf{r})}e^{il\varphi}. \tag{31}$$

In a rotating frame the energy functional of the system is

$$E_{\text{rot}}(l) = E(\phi_l) + \int d^3r(\phi_l^*)i\hbar\Omega\partial_\varphi(\phi_l). \tag{32}$$

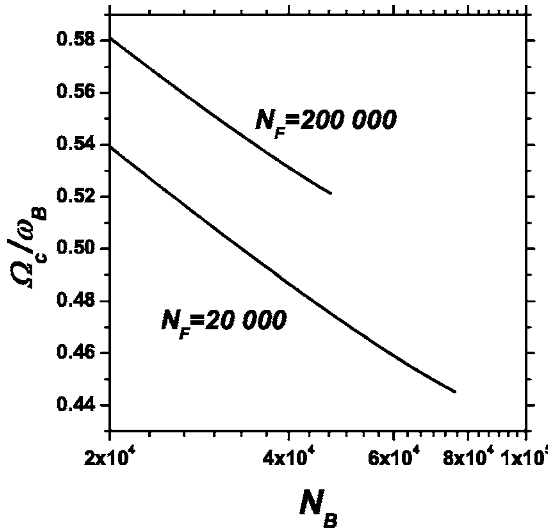

FIG. 7. Critical angular vortex velocity as a function of a logarithm of the number of bosons N_B in TFA.

After substituting the wave function for the vortex excitation (31) in Eq. (32), the effective confinement potential for the bosons becomes $l^2\hbar^2/2m\rho^2+V$, where $V=m\omega(\rho^2+\lambda^2z^2)/2$ and $\rho^2=x^2+y^2$. So within the TFA the density of the vortex state has the form

$$n_l(\mathbf{r}) = n(0)\left(1 - \sqrt{1 + \frac{\bar{\rho}^2 + \lambda^2\bar{z}^2 + \dfrac{l^2}{c_0\bar{\rho}^2} - R^2}{R_{\max}^2}}\right). \tag{33}$$

The important new qualitative feature of a vortex in the TFA is the appearance of a small hole of radius ξ, $\xi \propto l/R_{\max}$, but the remainder of the condensate density is essentially unchanged. The fractional change in the chemical potentials caused by the vortex $[\mu'(l)-\mu']/\mu'$ can be shown to be small [28,31], of the order of $1/N^{4/5}$. In the calculation of physical quantities involving the condensate density it is sufficient to retain the no-vortex density and simply cut off any divergent radial integrals at the appropriate core sizes $\xi=l/R_{\max}$. Note that using the unperturbed density for calculation of the vortex properties corresponds to the hydrodynamic limit.

In Fig. 5 the density distribution of bosons calculated from Eq. (33) is shown as a function of radius for different numbers of bosons and fermions.

The critical number of bosons in the presence of a vortex can be calculated from the condition that the expression under the square root in Eq. (33) is positive. This number is shown in Fig. 6 as a function of the number of fermions. As was expected the presence of the vortex increases the stability of the system against the collapse transition. Similar results were obtained in Ref. [19] by numerical solving of coupled mean-field equations for the boson and fermion densities. In their study, the instability signature is the failure of the numerical iterative process.

From Eqs. (32), (31), and (21)–(30) and $\Omega_c=[E_b(l=1)-E_b]/\hbar N_B$ one can find the critical angular velocity in TFA. The critical angular velocity as a function of the num-

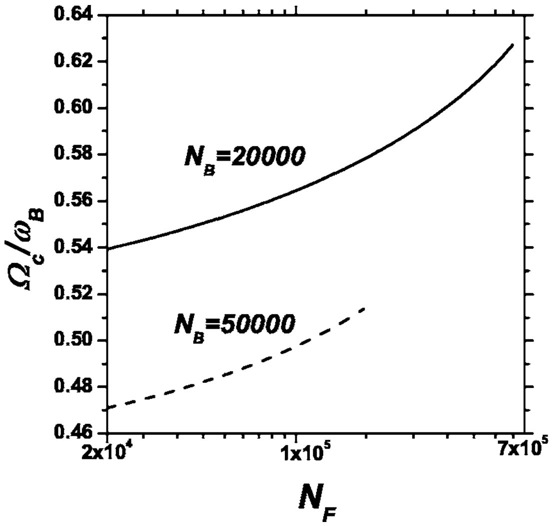

FIG. 8. Critical angular vortex velocity as a function of a logarithm of the number of fermions N_F in TFA.

bers of bosons and fermions correspondingly is shown in Figs. 7 and 8. One can see that the results are free from the mistakes which are present in the variational approach (see Figs. 3 and 4).

V. FINAL REMARKS

Finally, we make a short remark on the nature of the collapse transition. In this article we found the instability point of the Bose-Fermi mixture with attractive interaction between components with and without vortex. A strong rise of density of bosons and fermions [see Eq. (20)] in the collapsing condensate enhances intrinsic inelastic processes, in particular, the recombination in three-body interatomic collisions, as is the case for the well-known ^{7}Li condensates [26]. In the presence of a vortex there appears a hole in the middle of the condensate. This reduces the maximum density of the condensate (see Fig. 6) and increases the critical number of bosons. Recently Kagan and co-workers suggested the new microscopic mechanism of removing atoms from the system which is specific for the Bose-Fermi mixtures with attraction between components and is based on the formation of the boson-fermion bound states [33]. It seems that the description of the evolution of the collapsing condensate should include both these mechanisms.

ACKNOWLEDGMENTS

V.N.R. acknowledges valuable discussions with M. Yu. Kagan. The work was supported in part by the Russian Foundation for Basic Research (Grant No. 05-02-17280) and NWO-RFBR Grant No. 047.016.001. S.-T.C. is partly supported by a grant from NASA. N.M.C. acknowledges the support from RFBR (Grant No. 04-02-08159), Government Grant No. -4611.2004.2, and Russian Science Support Foundation.

[1] M. N. Anderson *et al.*, Science **269**, 198 (1995).
[2] K. B. Davis *et al.*, Phys. Rev. Lett. **75**, 3969 (1995).
[3] C. C. Bradley, C. A. Sackett, and R. G. Hulet, Phys. Rev. Lett. **78**, 985 (1997).
[4] B. DeMarco and D. S. Jin, Science **285**, 1703 (1999).
[5] A. G. Truscott *et al.*, Science **291**, 2570 (2001).
[6] K. M. O'Hara *et al.*, Science **298**, 2179 (2002).
[7] C. A. Regal, C. Ticknor, J. L. Bohn, and D. S. Jin, Nature (London) **424**, 47 (2003).
[8] M. Greiner, C. A. Regal, and D. S. Jin, Nature (London) **426**, 537 (2003).
[9] S. Jochim *et al.*, Science **302**, 2101 (2003).
[10] F. Schreck *et al.*, Phys. Rev. Lett. **87**, 080403 (2001).
[11] G. Modungo *et al.*, Science **297**, 2240 (2002).
[12] T. Miyakawa, T. Suzuki, and H. Yabu, Phys. Rev. A **64**, 033611 (2001).
[13] R. Roth, Phys. Rev. A **66**, 013614 (2002).
[14] S. T. Chui and V. N. Ryzhov, Phys. Rev. A **69**, 043607 (2004).
[15] S. T. Chui, V. N. Ryzhov, and E. E. Tareyeva, JETP Lett. **80**, 274 (2004).
[16] M. J. Bijlsma, B. A. Heringa, and H. T. C. Stoof, Phys. Rev. A **61**, 053601 (2000).
[17] M. Lewenstein, L. Santos, M. A. Baranov, and H. Fehrmann, Phys. Rev. Lett. **92**, 050401 (2004).
[18] D. M. Jezek, M. Barranco, M. Guilleumas, R. Mayol, and M. Pi, Phys. Rev. A **70**, 043630 (2004).
[19] Z. Hadzibabic *et al.*, Phys. Rev. Lett. **88**, 160401 (2002).
[20] M. W. Zwierlein, J. R. Abo-Shaeer, A. Schirotzek, C. H. Schunck, and W. Ketterle, Nature (London) **435**, 1047 (2005).
[21] J. Goldwin, S. B. Papp, B. DeMarco, and D. S. Jin, Phys. Rev. A **65**, 021402(R) (2002).
[22] V. N. Popov, *Functional Integrals in Quantum Field Theory and Statistical Physics* (Reidel, Dordrecht, 1983).
[23] H. T. C. Stoof, e-print cond-mat/9910441 (unpublished).
[24] D. A. Butts and D. S. Rokhsar, Phys. Rev. A **55**, 4346 (1997).
[25] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).
[26] Yu. Kagan, A. E. Muryshev, and G. V. Shlyapnikov, Phys. Rev. Lett. **81**, 933 (1998).
[27] Yu. Kagan, G. V. Shlyapnikov, and J. T. M. Walraven, Phys. Rev. Lett. **76**, 2670 (1996).
[28] A. L. Fetter and A. A. Svidzinsky, J. Phys.: Condens. Matter **13**, R135 (2001).
[29] S. T. Chui, V. N. Ryzhov and E. E. Tareyeva, JETP **91**, 1183 (2000).
[30] G. Baym and C. J. Pethick, Phys. Rev. Lett. **76**, 6 (1996).
[31] S. Sinha, Phys. Rev. A **55**, 4325 (1997).
[32] S. T. Chui, V. N. Ryzhov, and E. E. Tareyeva, Phys. Rev. A **63**, 023605 (2001).
[33] M. Yu. Kagan, I. V. Brodsky, D. V. Efremov, and A. V. Klaptsov, cond-mat/0209481 (unpublished).